Indian Statistical Institute
Second Semester Mid-Sem Examination 2005-2006
M.Math I Year
Functional Analysis

Time: 3 hrs | Date:27-02-06 | Max. Marks : 100

Remarks: Each question carries 20 marks.

1. Let (X, d) be a metric space. Let $C_b(X)$ denote the normed linear space consisting of all (real or complex valued) continuous functions on X, with usual operations and supremum norm.

   (a) Show that $C_b(X)$ is a Banach space.

   (b) Fix a point $x_0 \in X$. For any $x \in X$, let $\rho_x : \mathbf{X} \to \mathbb{R}$ be defined by $\rho_x(y) = d(x, y) - d(x_0, y),\ y \in X$. Show that $x \mapsto \rho_x$ is an isometric embedding of X in $C_b(X)$.

   (c) show that every metric space occurs as a dense subspace of a complete metric space.

2. Let $\mathbf{X}$ be a complex Banach space. Let $X_{\mathbb{R}}$ denote the same space, viewed as a real Banach space. Show that $f \mapsto$ Re (f) is an isometry from X^* onto $X^*_{\mathbb{R}}$.

3. (a) Prove that every non-empty closed and convex subset of a Hilbert space has a unique element of smallest norm.

   (b) Let C be the Banach space of all continuous function on [0,1] into $\mathbb{C}$, with supremum norm. Let $M = \{f \in C : \int_0^{1/2} f(t)dt - \int_{1/2}^{1} f(t)dt = 1\}$. Show that M is a closed and convex non-empty subset of C containing no element of smallest norm.

4. Let $K : [0,\ \infty) \times [0,\ \infty) \to I\!R$ be defined by $K(x,y) = \min(x,y)$.

   (a) Prove that K is an n.n.d. kernel. Let $\mathcal{H}$ denote the Hilbert space with reproducing kernel K.

   (b) Show that every element f of $\mathcal{H}$ is a continuous function with $f(0) = 0$.

   (c) Let $0 = x_0 < x_1 < x_2 < \ldots < x_n$ and $m_1, m_2, \ldots, m_n$ be real numbers. Let f be the unique continuous function on $[0, \infty)$ such that $f(0) = 0,\ f(x) = \text{constant for } x > x_n$, and $f\ |_{[x_{i-1}, x_i]}$ is a linear function of slope $m_i,\ \ 1 \leq i \leq n$. Show that $f \in \mathcal{H}$ and compute its norm.

5. Let $U : L^2(\mathbb{T}) \to L^2(\mathbb{T})$ ($\mathbb{T}$ = unit circle with normalised arc-length measure) be defined by $(Uf)(z) = zf(z),\ z \in \mathbb{T},\ f \in L^2(\mathbb{T})$. Prove that U is a unitary and compute its spectrum.